

# Path covering number and $L(2, 1)$ -labeling number of graphs \*

Changhong Lu and Qing Zhou

Department of Mathematics,  
East China Normal University,  
Shanghai 200241, P. R. China

March 2, 2013

## Abstract

A *path covering* of a graph  $G$  is a set of vertex disjoint paths of  $G$  containing all the vertices of  $G$ . The *path covering number* of  $G$ , denoted by  $P(G)$ , is the minimum number of paths in a path covering of  $G$ . An  $k$ - $L(2, 1)$ -labeling of a graph  $G$  is a mapping  $f$  from  $V(G)$  to the set  $\{0, 1, \dots, k\}$  such that  $|f(u) - f(v)| \geq 2$  if  $d_G(u, v) = 1$  and  $|f(u) - f(v)| \geq 1$  if  $d_G(u, v) = 2$ . The  $L(2, 1)$ -labeling number  $\lambda(G)$  of  $G$  is the smallest number  $k$  such that  $G$  has a  $k$ - $L(2, 1)$ -labeling. The purpose of this paper is to study path covering number and  $L(2, 1)$ -labeling number of graphs. Our main work extends most of results in [On island sequences of labelings with a condition at distance two, Discrete Applied Maths 158 (2010), 1-7] and can answer an open problem in [On the structure of graphs with non-surjective  $L(2, 1)$ -labelings, SIAM J. Discrete Math. 19 (2005), 208-223].

**Keywords.**  $L(2, 1)$ -labeling, Path covering number, Hole indexAlgorithm

## 1 Introduction

A *path covering* of a graph  $G$  is a set of vertex disjoint paths of  $G$  containing all the vertices of  $G$ . The *path covering number* of  $G$ , denoted by  $P(G)$ , is the minimum number of paths in a path covering of  $G$ . The *minimum path covering* of  $G$  is a path covering with size  $P(G)$ . The path covering problem is to find a minimum path covering of a graph. The path covering problem has received some alternative names

---

\*Supported in part by National Natural Science Foundation of China (No.10971248 ) and the Fundamental Research Funds for the Central Universities.

in the literature, such as optimal path cover [3, 28, 31] and path partition [32, 33]. It is evident that the path covering problem for general graphs is *NP*-complete since finding a path covering, consisting of a single path, corresponds directly to the Hamiltonian path problem. Polynomial-time algorithms to solve the path covering problem have been known for a few special classes of graphs, including trees [4, 16, 23, 27], block graphs [31, 33], interval graphs [3], circular-arc graphs [17], cographs [20], bipartite permutation graphs [28], cocomparability graphs [8] and distance-hereditary graphs [18]. The path covering problem has many practical applications in different areas, including mapping parallel programs to parallel architectures [24], code optimization [5] and program testing [25].

It is known that for a connected graph  $G$ , there is a spanning tree  $T$  of  $G$  such that  $P(G) = P(T)$  [4]. Hence, it is important to determine the path covering number of trees. There are polynomial-time algorithms to determine the path cover number of trees, surprisingly, however, almost no exact values for the path covering number of special families of trees are known.

The problem of vertex labeling with a condition at distance two, first studied by Griggs and Yeh [15], is a variation of the  $T$ -coloring problem introduced by Hale [?]. An  $L(2, 1)$ -labeling of a graph  $G$  is a mapping  $f$  from the vertex set  $V(G)$  to the set of all nonnegative integers such that  $|f(x) - f(y)| \geq 2$  if  $d_G(x, y) = 1$  and  $|f(x) - f(y)| \geq 1$  if  $d_G(x, y) = 2$ , where  $d_G(x, y)$  denotes the distance between the pair of vertices  $x, y$ . A  $k$ - $L(2, 1)$ -labeling is an  $L(2, 1)$ -labeling such that no label is greater than  $k$ . The  $L(2, 1)$ -labeling number  $\lambda(G)$  of  $G$  is the smallest number  $k$  such that  $G$  has a  $k$ - $L(2, 1)$ -labeling. A  $\lambda(G)$ - $L(2, 1)$ -labeling is referred to simply as a  $\lambda$ -labeling. It is shown that the  $L(2, 1)$ -labeling problem is *NP*-complete [14]. In general, it is hard to determine  $\lambda$  even for special graphs. The reader may consult [26, 30, 34, 35, 36] for known results on  $\lambda$ . The reader is referred to [6] for a survey and [7, 15, 29] for background information on this problem.

The following elegant result, proved by Georges, Mauro and Whittlesey [14], explored the relation between  $L(2, 1)$ -labeling problem and path covering problem.

**Theorem 1** (*cf. Theorem 1.1 in [14]*) Suppose that  $G$  is a graph of  $n$  vertices. Let  $G^c$  be the complement of  $G$ .

- (1)  $\lambda(G) \leq n - 1$  if and only if  $P(G^c) = 1$ ;
- (2)  $\lambda(G) = n + P(G^c) - 2$  if and only if  $P(G^c) \geq 2$ .

A  $k$ - $L(2, 1)$ -labeling is said to have a *hole*  $h$  with  $1 \leq h \leq k - 1$ , if the label  $h$  is not used. The minimum number of holes over all  $\lambda$ -labelings of a graph  $G$  is called the *hole index* of  $G$  and is denoted by  $\rho(G)$ . Several papers [1, 9, 10, 12, 13, 19, 21, 22] have studied  $\rho(G)$  and have investigated its connections with  $\lambda(G)$  and  $\Delta(G)$ , the maximum degree of  $G$ .

The following result by Georges and Mauro [12] established relation between  $\rho(G)$  and  $P(G^c)$ .

**Theorem 2** [12] *Let  $G$  be a graph on  $n$  vertices and  $\lambda(G) \geq n - 1$ . Then  $\rho(G) = P(G^c) - 1$ .*

It is not difficult to know that any two holes are non-consecutive in a  $\lambda$ -labeling. An *island* of a given  $\lambda$ -labeling of  $G$  with  $\rho(G)$  holes is a maximal set of consecutive integers used by the labeling. The *island sequence* is the ordered sequence of island cardinalities in nondecreasing order. Figure 1 [2] presents two different  $\lambda$ -labelings of the complete bipartite graph  $K_{2,3}$  with  $\lambda = 5$  and  $\rho = 1$ , and inducing the same island sequence (2, 3). Figure 2 presents two different  $\lambda$ -labelings of the non-connected graph  $K_5 \cup K_2$  with  $\lambda = 8$  and  $\rho = 2$ , and inducing two different island sequences (1, 1, 5) and (1, 3, 3), respectively. In [12], Georges and Mauro raised the following question to inquire about the existence of a connected graph possessing two  $\lambda$ -labelings with different island sequences.

**Question 3** [12] *Is there a connected graph admitting at least two distinct island sequences?*

A vertex in a graph is *heavy* if it has degree greater than 2, otherwise we say this vertex is *light*. A *heavy edge* of  $G$  is an edge incident to two heavy vertices of  $G$ . A *leaf* in a graph is a vertex of degree 1. A *vine* of a graph  $G$  is defined as a maximal path in  $G$  such that one endpoint is a leaf and each vertex in  $S$  is a light vertex in  $G$ . If  $G$  is not a path, then there is a unique heavy vertex adjacent to one of the

ends of  $S$ . We call such vertex the *center* of vine  $S$ . A *generalized star* is a tree which has exactly one heavy vertex and all its vines have the same number of vertices. A graph  $G$  is *2-sparse* if  $G$  contains no pair of adjacent vertices of degree greater than 2. The above notation is first introduced by Adams et al. [2] and they solved the above question by studying complements of 2-sparse trees.

**Theorem 4** (*cf. Theorem 2.8 in [2]*) *Let  $T$  be a 2-sparse tree. If  $T$  is neither a path nor a generalized star, then its complement  $T^c$  is connected and admits at least two different island sequences.*

They also determined the path covering number of 2-sparse trees.

**Theorem 5** (*cf. Theorem 2.4 in [2]*) *Let  $T$  be a 2-sparse tree with  $\ell \geq 2$  leaves. Then  $P(T) = \ell - 1$ .*

Furthermore, they determined the path covering number of more general connected non-cycle 2-sparse graphs.

**Theorem 6** (*cf. Theorem 3.2 in [2]*) *Let  $G$  be a connected 2-sparse graph with  $m \geq 1$  edges,  $n$  vertices, and  $\ell$  leaves. If  $G$  is not a cycle, then  $P(G) = \ell + m - n$ .*

Theorem 5 and Theorem 6 are important because it adds to the limited library of known path covering numbers. Also, when combined with the prior results in Theorem 1 and Theorem 2, it implies that the  $L(2, 1)$ -labeling number and hole index for complements of non-path 2-sparse trees and for complements of certain non-cycle 2-sparse graphs are determined. As pointed in [2], further study in two directions is encouraged:

- The existence of other families that admit multiple island sequence;
- These results in Theorem 5 and Theorem 6 are extended to include more general trees and graphs.

The main purpose of this paper is to extend the work in [2] through the investigation of the path covering number of trees and tree-like graphs. In Section 2, we give some lemmas which will be used in Sections 3, 4 and 5. In Section 3, we study the path covering numbers of trees and get some general results. In Section 4, we establish a

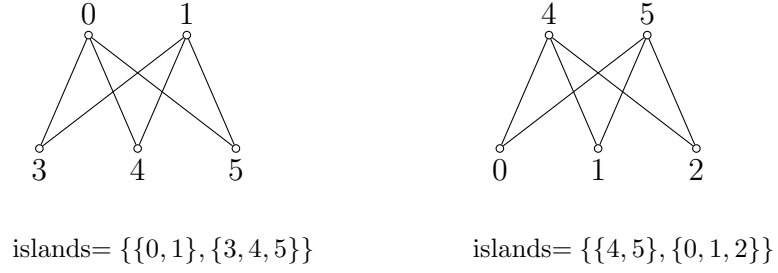


Figure 1 Two 5-labelings of  $K_{2,3}$  with 1 hole and island sequence  $(2, 3)$ .

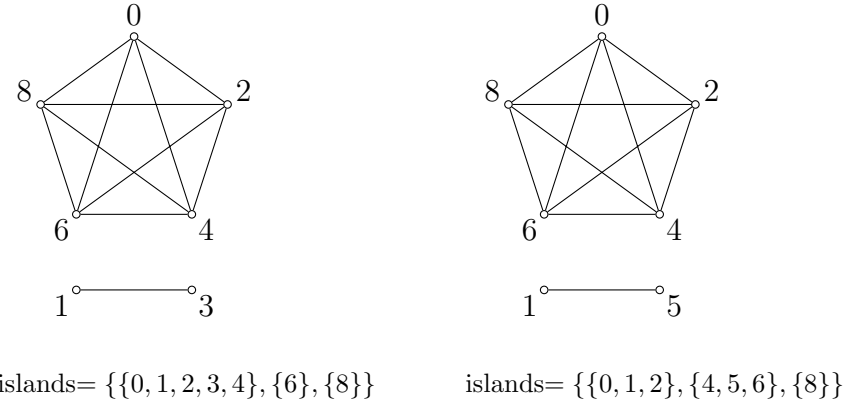


Figure 2 Two 8-labelings of  $K_5 \cup K_2$  with different island sequences.

characterization for trees, whose complements admit unique island sequences. Furthermore, a linear-time algorithm is given to determine whether the complement of a given tree  $T$  admits unique island sequence. In Section 5, we extend some results on trees to some families of graphs which contain 2-sparse graphs as subclass.

## 2 Preliminaries

If  $e$  is an edge of a graph  $G$ , then  $G - e$  is the graph obtained by deleting  $e$  from  $G$ . If  $H$  is a subgraph of  $G$ , the graph  $G - H$  is the graph obtained by deleting the vertices of  $H$  from  $G$  and any edge incident to a vertex in  $H$ . If  $f$  is an edge not in  $G$  but its ends are in  $G$  then  $G + f$  is the graph obtained by adding  $f$  to  $G$ . If two graphs  $G$  and  $H$  are disjoint, the graph  $G + H$  is defined as the graph with vertex and edge sets given respectively by the union of the vertex and edge sets of  $G$  and  $H$ .

The following result is obvious.

**Lemma 7** *Let  $S$  be a vine of a graph  $G$ . Then  $S$  is a subgraph of every minimum path covering of  $G$ .*

**Lemma 8** *If  $v$  is the common center of vines  $S_1$  and  $S_2$  in a graph  $G$ , then  $v$  is an internal vertex in every minimum path covering of  $G$ .*

**Proof.** Assume for contradiction that  $v$  is the end of path  $P$  in a minimum path covering of  $G$ . Since  $v$  is the center of both  $S_1$  and  $S_2$ , there exists a vine, say  $S_2$ , which is not contained in  $P$ . By Lemma 7,  $S_2$  must be a path in this path covering, and we call it  $Q$ . Let  $f$  be the edge incident to  $v$  and to one of the end of  $Q$ . Since  $P$  and  $Q$  are different,  $(P + Q) + f$  is a path. Replacing paths  $P$  and  $Q$  with  $(P + Q) + f$ , we obtain a path covering of  $G$  with smaller number of paths, a contradiction. ■

We now illustrate the *swapping construction* that is first introduced in [2]. Let  $P$  and  $Q$  be two different paths in a given path covering of a graph  $G$  such that  $P$  contains an edge  $e$  incident to an internal vertex  $v$  of  $P$ , and one end of  $Q$  is adjacent to  $v$  through an edge  $f$ . Clearly,  $P - e$  has two connected components, namely the paths  $P_1$  and  $P_2$  where  $v$  is an end of  $P_1$ . Since  $P$  and  $Q$  are different paths,  $(P_1 + Q) + f$  is a path. We can replace paths  $P$  and  $Q$  with paths  $(P_1 + Q) + f$

and  $P_2$  to obtain another path covering of  $G$  with the same number of paths. For convenience, we will say that this new path covering was obtained from the original one by *swapping  $e$  with  $f$* .

**Lemma 9** *Let  $v$  be a heavy vertex of a tree  $T$  and let  $e$  be an edge incident to  $v$ . Then  $e$  is not used in some minimum path covering of  $T$  and hence  $P(T) = P(T - e)$  if one of the following conditions holds:*

- (a)  *$v$  is the common center of at least three vines;*
- (b)  *$v$  is the common center of exactly two vines and  $e$  is not incident to any end of vines.*

**Proof.** Suppose that one of the above condition holds. Consider an arbitrary minimum path covering of  $T$ , by Lemma 8,  $v$  is an internal vertex in this minimum path covering. Assume that a path  $P$  in this path covering contains  $v$ . Then  $P$  contains exactly two edges incident to  $v$ . If  $e \notin E(P)$ , then  $e$  is contained in no paths within the given minimum path covering since  $v$  is an internal vertex in this minimum path covering. It is clear that  $P(T) = P(T - e)$ . Hence, we assume that  $e \in E(P)$ . In this situation, there exists a vine, say  $S$ , which is not contained in the path  $P$ , and by Lemma 7,  $S$  must be a path in this path covering, and we call it  $Q$ . Let  $f$  be the edge incident to  $v$  and to one of the end of  $Q$ . By swapping  $e$  with  $f$ , we obtain another minimum path covering of  $T$  and  $e$  is not contained in any path within the new path covering. It implies that  $P(T) = P(T - e)$ . ■

**Lemma 10** *If  $G$  is a graph such that each heavy vertex has at least three light neighbors, then  $P(G) = P(G - e)$ , where  $e$  is a heavy edge of  $G$ .*

**Proof.** Let  $e = uv$  be a heavy edge of  $G$ . Clearly,  $P(G - e) \geq P(G)$ . Consider an arbitrary minimum path covering of  $G$ , we will use this minimum path covering to construct a path covering of  $G - e$  with exactly  $P(G)$  paths, which would imply that  $P(G - e) = P(G)$ . Suppose that the path  $P$  contains the vertex  $u$  in this minimum path covering. If  $e$  is not in  $P$ , then this path covering of  $G$  is obviously a path covering of  $G - e$  with  $P(G)$  paths. So, we will consider the case when  $e$  is in  $P$ . Let  $u_1, \dots, u_k$  ( $k \geq 3$ ) be light vertices adjacent to  $u$  and let  $v_1, \dots, v_t$  ( $t \geq 3$ ) be light

vertices adjacent to  $v$ .  $P$  contains at most four vertices in  $\{u_1, \dots, u_k, v_1, \dots, v_t\}$ . Hence, there is a vertex, say  $u_3$ , not containing in  $P$ . Since  $u_3$  is a light vertex,  $u_3$  must be an end of one path, call it  $Q$ , in this minimum path covering of  $G$ . Let  $P_1$  be the connected component of  $P - e$  containing  $u$  and let  $P_2$  be the other component. Let  $f$  be the edge incident to  $u$  and  $u_3$ . Then  $P' = (P_1 + Q) + f$  is a path. Replacing  $P$  and  $Q$  with  $P'$  and  $P_2$  we obtain a path covering of  $G - e$  with  $P(G)$  paths. ■

### 3 Path covering number of trees

In this section, we establish some general results for the path covering number of trees.

**Theorem 11** *Let  $T$  be a tree with  $\ell$  leaves and  $h$  heavy edges. If  $T$  is not a single vertex, then*

$$\ell - h - 1 \leq P(T) \leq \ell - 1.$$

**Proof.** The proof will proceed by induction on the number of vertices in  $T$ . The result clearly holds when  $T$  is a star or  $T$  has exactly two vertices. Suppose now that  $T$  is other than a star and has at least three vertices. We can choose a vertex  $v \in V(T)$  with exactly one non-leaf neighbor  $u$  and  $k$  leaf neighbors  $z_1, \dots, z_k$ . It is known and easy to see that (PT1) and (PT2) hold.

(PT1) If  $k = 1$ , then  $P(T) = P(T')$ , where  $T' = T - z_1$ ;

(PT2) If  $k \geq 2$ , then  $P(T) = P(T') + k - 1$ , where  $T' = T - \{v, z_1, \dots, z_k\}$ .

If  $k = 1$ , then  $v$  becomes a leaf in  $T'$  and hence  $T'$  has  $\ell' = \ell$  leaves. So we have  $P(T) = P(T')$  (By (PT1)),  $\ell = \ell'$  and  $h = h'$ . By the induction hypothesis, we have  $\ell' - h' - 1 \leq P(T') \leq \ell' - 1$ . Hence,  $\ell - h - 1 \leq P(T) \leq \ell - 1$ . If  $k \geq 2$  and  $d_T(u) = 2$ , then  $u$  becomes a leaf in  $T'$  and hence  $T'$  has  $\ell' = \ell - k + 1$  leaves, we have  $P(T) = P(T') + k - 1$  (By (PT2)),  $\ell = \ell' + k - 1$  and  $h = h'$ , we can again apply the induction. If  $k \geq 2$  and  $d_T(u) \geq 3$ , we have  $P(T) = P(T') + k - 1$  (By (PT2)),  $\ell = \ell' + k$  and  $h \geq h' + 1$ . Applying the induction, we have  $\ell' - h' - 1 \leq P(T') \leq \ell' - 1$ . and hence  $\ell - h - 1 \leq P(T) \leq \ell - 1$ . ■



For a tree  $T$  with at least two vertices,  $T$  is 2-sparse if and only if  $h = 0$ . Hence, Theorem 5 (cf. Theorem 2.4 in [2]) is a direct corollary of Theorem 11. In fact we can get a stronger result for 2-sparse tree.

**Theorem 12** *Let  $T$  be a tree with  $\ell \geq 2$  leaves. Then  $T$  is 2-sparse if and only if  $P(T) = \ell - 1$ .*

**Proof.** *Sufficiency.* The result follows from Theorem 11.

*Necessity.* Suppose to the contrary that the number of heavy edges  $h \neq 0$ . Let  $e$  be a heavy edge of  $T$ .  $T'$  and  $T''$  are two components of  $T - e$ . It is clear that both  $T'$  and  $T''$  have at least three vertices since  $e$  is a heavy edge of  $T$ . Assume that  $T'$  ( $T''$ , respectively) has  $\ell'$  ( $\ell''$ , respectively) leaves. By Theorem 11,  $P(T') \leq \ell' - 1$  and  $P(T'') \leq \ell'' - 1$ . Hence,

$$\ell - 1 = P(T) \leq P(T') + P(T'') \leq (\ell' + \ell'') - 2.$$

But  $\ell' + \ell'' = \ell$  as  $e$  is a heavy edge of  $T$ . Therefore,  $\ell - 1 = P(T) \leq \ell - 2$ . This is a contradiction. So,  $h = 0$  and  $T$  is 2-sparse. ■

Now we extend partial results in Theorem 12 to include more general trees.

**Theorem 13** *Let  $T$  be a tree with  $\ell \geq 2$  leaves in which each heavy vertex has at least one light neighbors. Let  $S$  be the subset of  $V(T)$  such that each element of  $S$  is a heavy vertex with exactly one light neighbor. Then*

$$P(T) = \ell - h + s - t - 1,$$

*where  $h$  is the number of heavy edges and  $s$  is the size of  $S$  and  $t$  is the size of a maximum matching of the subgraph induced by  $S$ .*

**Proof.** The proof will proceed by induction on the number of vertices in  $T$  and use the same method in the proof of Theorem 11. When  $T$  is a star or  $T$  has exactly two vertices, the result clearly holds. Suppose now that  $T$  is a tree other than a star and has at least three vertices. We can also choose a vertex  $v \in V(T)$  with exactly one non-leaf neighbor  $u$  and  $k$  leaf neighbors  $z_1, \dots, z_k$ . Let  $S'$  be the subset of  $T'$  such

that each element of  $S'$  is a heavy vertex with exactly one light neighbor in  $T'$  and  $t'$  be the size of a maximum matching of the subgraph induced by  $S'$ . By induction,  $P(T') = \ell' - h' + s' - t' - 1$ , where  $\ell'$  ( $h'$ , respectively) is the number of leaves (heavy edges, respectively) and  $s'$  is the size of  $S'$  in  $T'$ .

If  $k = 1$ , then  $v$  becomes a leaf in  $T'$  and hence  $T'$  has  $\ell' = \ell$  leaves. So we have  $P(T) = P(T')$  (By (PT1)),  $\ell = \ell'$ ,  $h = h'$ ,  $s = s'$  and  $t' = t$ . Applying induction, we clearly have  $P(T) = \ell - h + s - t - 1$ . Now, we assume that  $k \geq 2$ .

If  $d_T(u) = 2$ , then  $\ell' = \ell - k + 1$ ,  $h' = h$ ,  $s' = s$  and  $t = t'$ . Hence,  $P(T) = P(T') + k - 1 = \ell - h + s - t - 1$  as desired. If  $d_T(u) = 3$  with  $u \notin S$  or  $d_T(u) \geq 4$ , then  $\ell' = \ell - k$ ,  $h' = h - 1$ ,  $s' = s$  and  $t' = t$ . Hence,  $P(T) = P(T') + k - 1 = \ell - h + s - t - 1$  as desired. Now we assume that  $u$  has three neighbors  $v$ ,  $u_1$  and  $u_2$  and  $u \in S$ . Without loss of generality, we assume that  $u_1$  is a light vertex and  $u_2$  is a heavy vertex in  $T$ . If  $u_2 \notin S$ , then  $\ell' = \ell - k$ ,  $h' = h - 2$ ,  $s' = s - 1$  and  $t' = t$ . Hence,  $P(T) = P(T') + k - 1 = \ell - h + s - t - 1$  as desired. Now we consider the case  $u_2 \in S$ . In this situation,  $d_{T'}(u) = 2$ ,  $uu_2$  is not a heavy edge in  $T'$  and hence  $u_2 \notin S'$ . Thus  $\ell' = \ell - k$ ,  $h' = h - 2$  and  $s' = s - 2$ . Let  $C$  denote the connected component containing  $u$  and  $u_2$  in the subgraph induced by  $S$ . Note that  $C$  is a tree in which  $u$  is a leaf. Let  $C' = C - uu_2$ . So, it is easy to know that the size of a maximum matching in  $C'$  than that in  $S$  is small one. So  $t' = t - 1$  and Hence  $P(T) = P(T') + k - 1 = \ell - h + s - t - 1$  as desired. ■

We say a graph is *general 2-sparse* if each its heavy vertex has at least two light neighbors. Obviously, a 2-sparse graph must be general 2-sparse. Furthermore, we have

**Theorem 14** *Let  $T$  be a tree with  $\ell \geq 2$  leaves and  $h$  heavy edges. Then  $T$  is general 2-sparse if and only if  $P(T) = \ell - h - 1$ .*

**Proof.** *Sufficiency.* It follows from Theorem 13 by using  $s = 0$  and  $t = 0$ .

*Necessity.* We use induction on  $h$ . If  $h = 0$ , then  $P(T) = \ell - 1$  and hence it is ok by Theorem 12. We assume that  $h \neq 0$ . Let  $r$  be an arbitrary vertex of  $T$  and let  $e = uv$  be a heavy edge of  $T$  such that the distance between  $r$  and  $e$  is largest. Furthermore, we assume that  $d_T(r, u) = d_T(r, v) - 1$ . By the selection of  $e$ , all neighbors of  $v$

except that  $u$ , say  $v_1, \dots, v_k$  ( $k \geq 2$ ), are ends of vines. By Lemma 9,  $e$  is not used in some minimum path covering of  $T$ , and hence  $P(T) = P(T - e)$ . Suppose that  $T'$  ( $T''$ , respectively) is the connected component of  $T - e$  containing  $u$  ( $v$ , respectively). It is clear that both  $T'$  and  $T''$  have at least three vertices. Assume that  $T'$  ( $T''$ , respectively) has  $\ell'$  ( $\ell''$ , respectively) leaves and  $h'$  ( $h''$ , respectively) heavy edges. By Theorem 11,  $P(T') \geq \ell' - h' - 1$  and  $P(T'') \geq \ell'' - h'' - 1$ . Note that  $\ell' + \ell'' = l$  and  $h' + h'' + 1 \leq h$ . Hence,

$$\ell - h - 1 = P(T) = P(T') + P(T'') \geq \ell' - h' - 1 + \ell'' - h'' - 1 \geq \ell - h - 1.$$

This implies that  $P(T') = \ell' - h' - 1$ ,  $P(T'') = \ell'' - h'' - 1$  and  $h' + h'' + 1 = h$  hold together. By induction hypothesis, we have both  $T'$  and  $T''$  are general 2-sparse, and  $T$  is obtained from  $T'$  and  $T''$  by adding an edge  $e = uv$ . It is obvious that the vertex  $v$  has least two light neighbors. If  $d_{T'}(u) \geq 3$ , it is clear that any heavy vertex in  $T'$  has at least two light neighbors when  $e$  is added. We assume that  $d_{T'}(u) = 2$  and  $u_1, u_2$  are its neighbors in  $T'$ . We claim that  $u_1$  and  $u_2$  are both light vertices. If not, without loss of generality, we assume that  $u_1$  is a heavy vertex in  $T$ . Let  $f = uu_1$ . Then  $f$  is a heavy edge in  $T$ , but it is not a heavy edge in  $T'$  as  $d_{T'}(u) = 2$ . So,  $h' + h'' \leq h - 2$ , it contracts with  $h' + h'' + 1 = h$ . Therefore,  $u_1$  and  $u_2$  are both light vertices in  $T$ , and hence  $T$  is general 2-sparse. ■

The following result is a direct corollary of Theorem 1, Theorem 2 and Theorem 13.

**Corollary 15** *Let  $T$  be a non-path tree satisfying the conditions in Theorem 13. Then  $\lambda(T^c) = n + \ell - h + s - t - 3$  and  $\rho(T^c) = \ell - h + s - t - 2$ , where  $n$  ( $\ell$ ,  $h$ , respectively) is the number of vertices (leaves, heavy edges, respectively) and  $s$  ( $t$ , respectively) is the size of  $S$  (a maximum matching of the subgraph induced by  $S$ , respectively).*

At the end of this section, we remark that it would be interesting to investigate the path covering number of other more general families of trees.

## 4 Island sequences for complements of trees

Given a minimum path covering  $\mathcal{P}$  of a graph  $G$ , the *path sequence* of  $\mathcal{P}$  is the ordered sequence of the numbers of vertices of paths in  $\mathcal{P}$  in nondecreasing order (note that this definition allows for repeated cardinalities). As shown in [14], if  $P(G) \geq 2$ , then a minimum path covering  $\mathcal{P}$  of a graph  $G$  can induce a  $\lambda(G^c)$ -labeling  $f_{\mathcal{P}}$  of  $G^c$  with  $P(G) - 1$  holes and the island sequence of  $f_{\mathcal{P}}$  is same as the path sequence of  $\mathcal{P}$ . We refer readers to [14] for the complete proof. Hence,  $G^c$  admits multiple island sequences if and only if  $G$  admits distinct path sequences and  $P(G) \geq 2$ . In this section, we will establish a constructive characterization for trees with unique path sequence.

A *labeled generalized star* is a generalized star in which all neighbors of its center are labeled  $B$  and other non-leaf vertices are labeled  $A$ . Figure 3 (a) illustrates a labeled generalized star with three vines of length 3. For convenience, a path  $P$  of length  $2k$  is called a labeled generalized star with two vines of length  $k - 1$  in which two neighbors of its center are labeled  $B$  and other non-leaf vertices are labeled  $A$ . Figure 3 (b) illustrates a labeled generalized star with two vines of length 3. A *labeled path* is a path with at least three vertices in which all non-leaf vertices are labeled  $A$ . Figure 3 (c) illustrates a labeled path of length 5.

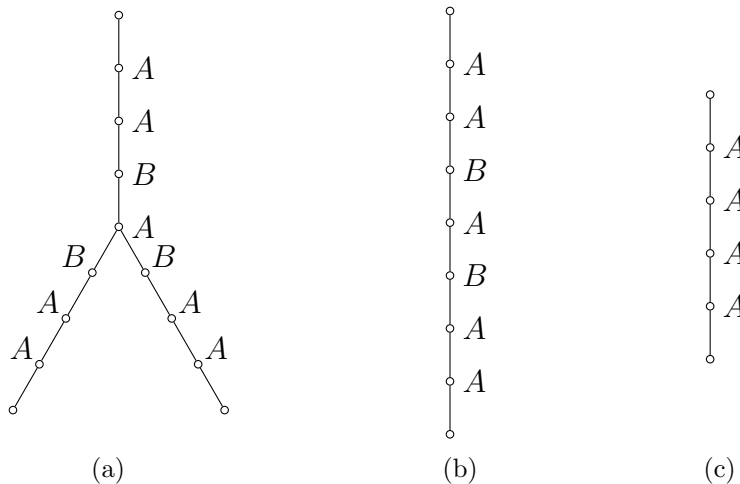


Figure 3

To state the constructive characterization of trees with unique path sequence, we

need to introduce a family of labeled trees and three types of operations.

A family of labeled trees  $\mathcal{F}$  is defined as  $\mathcal{F} = \{T \mid T \text{ is obtained from a labeled generalized star with at least three vines or a labeled path by a finite sequence of operations of Type-1, Type-2 or Type-3}\}$ .

Let  $T \in \mathcal{F}$  be a labeled tree in which all non-leaf vertices are labeled  $A$  or  $B$ .

**Type-1 operation:** Attach a labeled generalized star  $S$  with at least three vines to  $T$  by adding an edge  $uv$ , where  $u$  is a vertex in  $T$  labeled  $A$  and  $v$  is the center of  $S$ . Figure 4 (a) illustrates this operation.

**Type-2 operation:** Attach a labeled path  $P$  to  $T$  by adding an edge  $uv$ , where  $u$  is a vertex in  $T$  labeled  $A$  and  $v$  is a non-leaf vertex in  $P$ . Figure 4 (b) illustrates this operation.

**Type-3 operation:** Attach a labeled generalized star  $S$  to  $T$  by adding an edge  $uv$ , where  $v$  is the center of  $S$  and  $u$  is a vertex in  $T$  labeled  $B$  such that the length of each vine in  $S$  is same as that of the labeled generalized star  $S'$ , which is the original/attached labeled generalized star containing  $u$ . Figure 4 (c) illustrates this operation.

**Theorem 16** *Every labeled tree in  $\mathcal{F}$  admits unique path sequence.*

**Proof.** Suppose  $T$  is a labeled tree in  $\mathcal{F}$ . Let  $s(T)$  denote the number of operations required to construct  $T$ . We need prove a more general statement by induction on  $s(T)$  as follows:

- (C1)  $T$  admits unique path sequence;
- (C2) Every vertex labeled  $A$  is an internal vertex in any minimum path covering of  $T$ ;
- (C3) For each vertex labeled  $B$ , if it is one end of a path in some minimum path covering of  $T$ , then the path is exactly the vine, containing this vertex, of the labeled generalized star in the construction process of  $T$ .

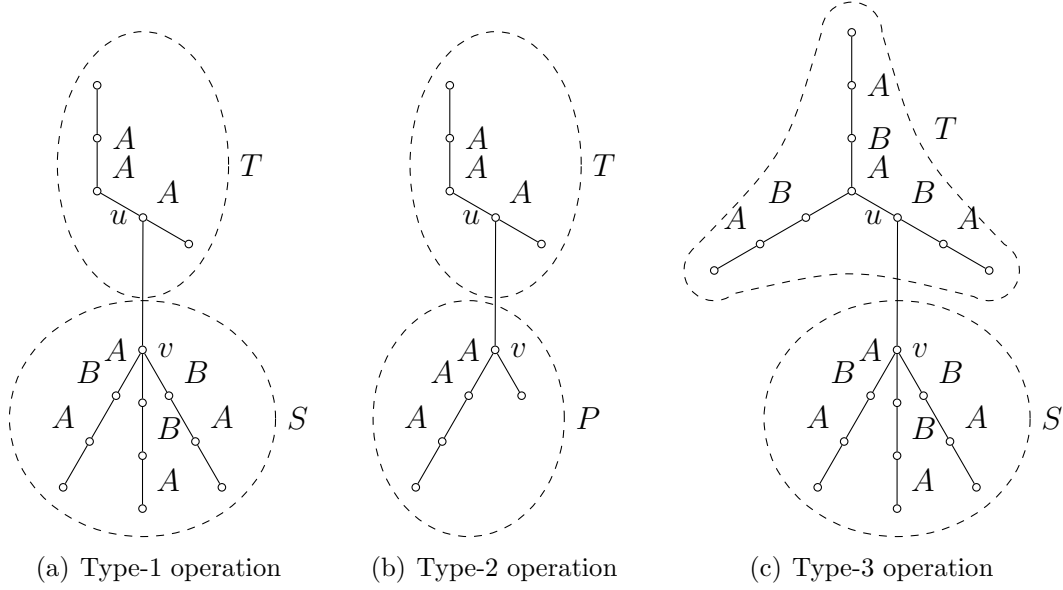


Figure 4

If  $s(T) = 0$ , then  $T$  is a labeled generalized star with at least three vines or a labeled path. It is obvious that  $T$  satisfies (C1)-(C3), and hence the assertion is true.

Assume that  $T'$  satisfies (C1)-(C3) for all trees  $T' \in \mathcal{F}$  with  $s(T') < k$ , where  $k \geq 1$  is an integer. Let  $T$  be a tree with  $s(T) = k$ . Then  $T$  is obtained from a tree  $T'$  by one of a Type-1, a Type-2 or a Type-3 operation, and  $T' \in \mathcal{F}$ . Applying the inductive hypothesis to  $T'$ ,  $T'$  satisfies (C1)-(C3).

If  $T$  is obtained from a tree  $T'$  by a Type-1 operation, by Lemma 9,  $P(T) = P(T') + \ell - 1$ , where  $\ell \geq 3$  is the number of leaves in the generalized star  $S$ . Suppose that vertices labeled  $B$  in  $S$  are  $v_1, \dots, v_\ell$ . Let  $e = uv$  and  $e_i = vv_i$  for  $i = 1, \dots, \ell$ . Given a minimum path covering of  $T$ , by Lemma 8,  $v$  is an internal vertex in a path  $P$ . If  $P$  contains  $e$ , by swapping  $e$  with some  $e_i$ , we obtain another minimum path covering of  $T$ , which also induces a minimum path covering of  $T'$  and  $e$  is not used. Meanwhile,  $u$  is one of ends of a path in this minimum path covering of  $T'$ . It contradicts with the fact  $T'$  satisfies (C2), i.e., every vertex in  $T'$  labeled  $A$  is an internal vertex in every minimum path covering of  $T'$ . Hence,  $e = uv$  is not used in any minimum path covering of  $T$ . Since both  $T'$  and  $S$  satisfy (C1),  $T$  admits unique path sequence, i.e.,  $T$  satisfies (C1). By induction hypothesis, every vertex in  $T'$  clearly satisfies (C2)-(C3). Lemma 7 implies that every vertex labeled  $A$  in  $S$

satisfies (C2). Let  $P$  be the path containing the vertex  $v$  in a minimum path covering of  $S$ . Without loss of generality, we assume that  $e_1 \in E(P)$  and  $e_2 \in E(P)$ . Then the vine containing  $v_i$  in  $S$  ( $3 \leq i \leq \ell$ ) is exactly a path in this minimum path covering of  $S$ . By swapping  $e_1$  ( $e_2$ , respectively) with  $e_\ell$ , we can get a new minimum path covering of  $S$  such that the vine containing  $v_1$  ( $v_2$ , respectively) in  $S$  is exactly a path in this minimum path covering. Therefore, every vertex in  $S$  satisfies (C3).

If  $T$  is obtained from a tree  $T'$  by a Type-2 operation, we can use similar arguments as above and get that  $e = uv$  is not used in any minimum path covering of  $T$ , and hence the labeled path  $P$  attached into  $T'$  is exactly a path in any minimum path covering of  $T$ . It is easy to check that  $T$  satisfies (C1)-(C3) in this situation.

If  $T$  is obtained from a tree  $T'$  by a Type-3 operation, Lemma 9 implies that  $P(T) = P(T') + \ell - 1$ , where  $\ell$  is the number of leaves in the labeled generalized star  $S$ . By induction hypothesis, both  $T'$  and  $S$  have unique path sequence. Suppose that the unique path sequence of  $T'$  is  $(x_1, \dots, x_{P(T')})$ . Obviously, the unique path sequence of  $S$  is  $(2s_1 + 1, s_1, \dots, s_1)$ , where the length of path sequence of  $S$  is  $\ell - 1$  and  $s_1$  is the number of vertices in each vine of  $S$ . It is clear that  $(x_1, \dots, x_{P(T')}, s_1, \dots, s_1, 2s_1 + 1)$  is a path sequence of  $T$  if we do not consider nondecreasing order. When the edge  $uv$  is not used in a minimum path covering of  $T$ , its path sequence is clearly  $(x_1, \dots, x_{P(T')}, s_1, \dots, s_1, 2s_1 + 1)$  if the nondecreasing order is not considered. Hence we assume that  $uv$  is used in a minimum path covering  $\mathcal{P}$  of  $T$  and  $P \in \mathcal{P}$  is the path containing the edge  $uv$  and with  $p$  vertices. By Lemmas 7 and 8, there is a vine of  $S$  as a path in  $\mathcal{P}$ . By swapping the edge  $uv$  and the edge  $wv$ , where  $w$  is one of ends of this vine, we get a new minimum path covering  $\mathcal{P}'$ , in which the edge  $uv$  is not used. In  $\mathcal{P}'$ ,  $v$  is an end of a path, say  $P'$ , with  $p - (s_1 + 1)$  vertices. As  $\mathcal{P}'$  restricted to the tree  $T'$  induce a minimum path covering of  $T'$ , by induction,  $P'$  is exactly the vine containing  $v$  in the labeled generalized star. Type-3 operation implies that  $p - (s_1 + 1) = s_1$ . Hence,  $p = 2s_1 + 1$ . Since both  $T'$  and  $S$  admit the unique path sequence, the path sequence induced by  $\mathcal{P}'$  is  $(x_1, \dots, x_{P(T')}, s_1, \dots, s_1, 2s_1 + 1)$  if the nondecreasing order is not considered. Note that  $\mathcal{P}$  can also be obtained from  $\mathcal{P}'$  by swapping  $wv$  and  $uv$ . Hence,  $\mathcal{P}$  admits the same path sequence with  $\mathcal{P}'$ . Therefore,  $T$  satisfies (C1).

Let  $w$  be a vertex in  $T$  labeled  $A$ . If  $w$  is a vertex in  $S$ , Lemmas 7 and 8 implies that  $w$  is an internal vertex in every minimum path covering of  $T$ . We assume that  $w \in V(T')$ . Let  $\mathcal{P}$  be a minimum path covering of  $T$  in which  $w$  is an end of a path. Without loss of generality, we assume that the edge  $uv$  is used in  $\mathcal{P}$ , and hence there is a path  $P \in \mathcal{P}$  containing  $uv$ . Similarly, by swapping  $uv$  and an edge incident to  $v$ , we can get a new minimum path covering  $\mathcal{P}'$  of  $T$  in which  $uv$  is not used. Hence,  $\mathcal{P}'$  restricted to  $T'$  induces a minimum path covering of  $T'$ . Since  $w$  is an end of a path in  $\mathcal{P}$ , then  $w$  is also an end of a path in this minimum path covering of  $T'$ . However, by induction hypothesis,  $T'$  satisfies (C2). It is a contradiction. Therefore,  $w$  must be an internal vertex in  $\mathcal{P}$ , and hence  $T$  satisfies (C2).

By Lemma 9, any minimum path covering of  $T'$  and any minimum path covering of  $S$  consists of a minimum path covering of  $T$ . By induction hypothesis, both  $T'$  and  $S$  satisfy (C3). Hence, it is easy to see that  $T$  satisfies (C3).  $\blacksquare$

**Theorem 17** *Let  $T$  be a tree with at least three vertices. If  $T$  admits unique path sequence, then we can labeled  $T$  by  $A$  and  $B$  such that it is an element of  $\mathcal{F}$ .*

**Proof.** Suppose  $T$  is a tree with at least three vertices and it admits unique path sequence. We use induction on  $hv(T)$ , the number of heavy vertices in  $T$ , to prove a more general statement as follows:

- (D1)  $T$  can be labeled by  $A$  and  $B$  such that it is an element of  $\mathcal{F}$ ;
- (D2) each vertex labeled  $A$  is an internal vertex in every minimum path covering of  $T$ ;
- (D3) each vertex labeled  $B$  is an end of the path, induced by the vine containing this vertex in the labeled generalized star, in some minimum path covering of  $T$ .

The assertion is clearly true if  $hv(T) = 0, 1$ . We now assume that  $hv(T) \geq 2$  and the assertion holds for smaller values of  $hv(T)$ .

Let  $r$  be an arbitrary vertex in  $T$  and let  $v$  be the heavy vertex such that  $d_T(r, v)$  is largest. Then there is a unique vertex, say  $u$ , with  $d_T(r, u) = d_T(r, v) - 1$ . Let  $e$  be the edge incident to  $u$  and  $v$ . By the selection of  $v$ ,  $v$  has at least two neighbors different from  $u$ , each of which is one end of a vine. By Lemma 9,  $e$  is not used in some minimum path covering of  $T$  and hence  $P(T) = P(T - e) = P(T') + P(T'') = P(T') + d_T(v) - 2$ ,



where  $T'$  ( $T''$ , respectively) is the component containing  $u$  ( $v$ , respectively) of  $T - e$ . It is easy to see that  $T'$  admits unique path sequence since otherwise  $T$  admits different path sequences. Hence, by induction hypothesis, (D1)-(D3) hold for the tree  $T'$ . Suppose that neighbors of  $v$  are  $u, v_1, \dots, v_k$ , where  $k = d_T(v) - 1 \geq 2$ . Let  $S_i$  be the vine containing  $v_i$  and  $e_i = vv_i$  for  $i \in \{1, 2, \dots, k\}$ .

*Case 1.  $u$  is labeled  $A$  in the label tree  $T'$ .*

It implies that  $u$  is an internal vertex in every minimum path covering of  $T'$ . If there is a path  $P$  containing  $e$  in a minimum path covering of  $T$ , then by swapping  $e$  and  $e_i$ , where  $e_i$  is not contained in  $P$ , we get a new minimum path covering of  $T$ , in which  $e$  is not used. Restricted to  $T'$ , this new minimum path covering induces a minimum path covering of  $T'$  such that  $u$  is an end of a path. It contradicts that  $u$  is an internal vertex in every minimum path covering of  $T'$ . Therefore,  $e$  is not used in any minimum path covering of  $T$ .

If  $k = 2$ , then  $T''$  is a path with at least three vertices, and hence we label each non-leaf vertex of this path by  $A$  and  $T$  is obtained from  $T'$  by one Type-2 operation. It is easy to check that (D1)-(D3) hold for  $T$  in this situation.

If  $k \geq 3$ , we claim that  $T''$  is a generalized star. Suppose to the contrary that there are two vines, say  $S_1$  and  $S_2$ , whose lengths are different. Vines  $S_2, S_4, \dots, S_k$  and the path induced by  $S_1 \cup S_3 \cup \{v\}$  form a minimum path covering of  $T''$ . Similarly, vines  $S_1, S_4, \dots, S_k$  and the path induced by  $S_2 \cup S_3 \cup \{v\}$  form another minimum path covering of  $T''$ . Obviously, path sequences of these two minimum path covering of  $T''$  are different as  $S_1$  and  $S_2$  have different length, i.e.,  $T''$  admits different path sequences, and hence  $T$  admits different path sequences, a contradiction. So, all vines in  $T''$  have the same number of vertices and  $T''$  is a general star. Then we label vertices  $v_1, \dots, v_k$  by  $B$  and label other non-leaf vertices in  $T''$  by  $A$ , and hence  $T$  is obtained from  $T'$  by one Type-1 operation. It is easy to check that (D1)-(D3) hold for  $T$  in this situation since  $e$  is not used in any minimum path covering of  $T$ .

*Case 2.  $u$  is labeled  $B$  in the label tree  $T'$ .*

Since  $u$  is labeled  $B$  in the label tree  $T'$ ,  $u$  is an end of the path, induced by the vine (say  $S$ ) containing  $u$  in the labeled generalized star, in some minimum path

covering  $\mathcal{P}'$  of  $T'$ . Assume that there is a vine in  $T''$ , say  $S_1$ , having different length with  $S$ . Let  $\mathcal{P}''$  be a minimum path covering of  $T''$  such that  $S_1 \cup S_2 \cup \{v\}$  forms a path in  $\mathcal{P}''$ . As  $P(T) = P(T') + P(T'')$ ,  $\mathcal{P}' \cup \mathcal{P}''$  is a minimum path covering of  $T$ . Now by swapping  $e_1$  and  $e$ , we get a new path covering of  $T$  in which  $S \cup S_2 \cup \{v\}$  forms a path. It is easy to find that it has different path sequence with  $\mathcal{P}' \cup \mathcal{P}''$ , a contradiction. Hence, both  $S_i$  and  $S$  have the same number of vertices for all  $i \in \{1, 2, \dots, k\}$ . We now label  $T''$  as follows: all neighbors of  $v$  in  $T''$  are labeled by  $B$  and other non-leaf vertices in  $T''$  are labeled by  $A$ . Therefore,  $T''$  is a labeled generalized star and  $T$  is obtained from  $T'$  by one Type-3 operation. Using the swapping construction and Lemmas 7, 8, 9, it is straightforward to check that (D2)-(D3) hold for  $T$ . ■

Theorem 16 and Theorem 17 tell us that a tree with at least three vertices admits unique path sequence if and only if it is an underlying tree in  $\mathcal{F}$  (An underlying tree in  $\mathcal{F}$  is a tree in  $\mathcal{F}$  deleted its labels). Recall the discussion in the begin of this section, we immediately obtain a clear characterization for trees, whose complements admit multiple island sequences.

**Theorem 18** *Let  $T$  be a tree which is neither a path nor a generalized star. Then  $T^c$  admits multiple island sequences if and only if  $T$  is not an underlying tree in  $\mathcal{F}$ .*

As we know, if a tree  $T$  is neither a star nor a path, then  $T^c$  is connected. Hence, Theorem 18 settles Question 3 posed by Georges and Mauro [12]. It is obvious that 2-sparse trees are not underlying trees in  $\mathcal{F}$ . Therefore, Theorem 4 (cf. Theorem 2.8 in [2]) is a corollary of Theorem 18.

Based on Theorem 18 and Lemmas 8, 9, we have the following algorithm to determine whether its complement of a given tree  $T$  has unique island sequence.

**Algorithm DUIS.** Determine whether the complement  $T^c$  has unique island sequence for a given tree  $T$ .

**Input.** A tree  $T$ .

**Method.**

**Step 0:** Initializes the labels of all vertices  $v \in V(T)$  with  $f(v) = 0$  and  $\ell(v) = 0$ ;

**Step 1:** If  $T$  is a path  $P_k$  with  $k \geq 1$ , then  
     If there is a leaf or isolated vertex  $x$  with  $\ell(x) = A$  or  $0 < f(x) \neq k$ , then  
         go to **Step 7**;  
     else  
         Output “ $T^c$  has unique island sequence”, and stop.  
     endif  
 endif

**Step 2:** Let  $r$  be a leaf of  $T$ .  $v$  is the heavy vertex such that  $d_T(v, r)$  is the largest and  $u$  is the vertex with  $d_T(u, r) = d_T(v, r) - 1$ . let  $e = uv$  and let  $T'$  ( $T''$ , respectively) be the connected component of  $T - e$  containing  $u$  ( $v$ , respectively).

**Step 3:** If  $T''$  is neither a path nor a generalized star, then  
     go to **Step 7**;  
 endif

**Step 4:** If  $T''$  is a generalized star  $S$ , let  $k$  be the number of vertices of a vine of  $S$ .  
     If (There is a neighbor  $x$  of  $v$  in  $S$  with  $\ell(x) = A$  or  $0 < f(x) \neq k$ )  
     or (There is a leaf  $x$  in  $S$  with  $\ell(x) = A$  or  $f(x) > 0$ ), then  
         go to **Step 7**;  
     endif  
     If  $0 < f(u) \neq k$ , then  
          $\ell(u) := A$ ;  
          $f(u) := 0$ ;  
     endif  
     If  $f(u) = 0$  and  $\ell(u) = O$ , then  
          $f(u) := k$   
     endif  
      $T := T'$ ;  
     go back to **Step 1**;  
 endif

**Step 5:** If  $T''$  is a path  $P_k(k \geq 4)$  such that  $v$  is not the middle vertex in  $P_k$ , then  
 If there is a leaf  $x$  in  $T''$  with  $\ell(x) = A$  or  $0 < f(x) \neq k$ , then  
     go to **Step 7**;  
 else  
      $\ell(u) := A$ ;  
      $f(u) := 0$ ;  
      $T := T'$   
     go back to **Step 1**;  
 endif  
 endif  
**Step 6:** If  $T''$  is a path  $P_k(k \geq 3)$  such that  $v$  is the middle vertex in  $P_k$ ,  
 let  $v_1$  and  $v_2$  be two neighbors of  $v$  in  $T''$ ,  
 If there is a leaf  $x$  in  $T''$  with  $\ell(x) = A$  or  $0 < f(x) \neq k$ , then  
     go to **Step 7**;  
 endif  
 If (There is a vertex  $x \in \{u, v_1, v_2\}$  with  $\ell(x) = A$  or  $0 < f(x) \neq \frac{k-1}{2}$ ) or (There is a leaf  $x$  in  $T''$  with  $f(x) = k$ ), then  
      $\ell(u) := A$ ;  
      $f(u) := 0$ ;  
 else  
      $f(u) := \frac{k-1}{2}$ ;  
 endif  
 $T := T'$ ;  
 go back to **Step 1**;  
 endif  
**Step 7:** Output “ $T^c$  has multiple island sequences”, and stop.

In **Algorithm DUIS**, two labels on each vertex  $v$ , denoted by  $\ell(v)$ ,  $f(v)$ , are used. For convenience, we first initializes the labels with  $\ell(v) = O$  and  $f(v) = 0$  for all vertices  $v \in V(T)$ . In Steps 4, 5, 6 of **Algorithm DUIS**, when the value  $A$  is assign to  $\ell(u)$ , it implies that  $u$  must be an internal vertex in any minimum path covering of  $T'$  if  $T$  has unique path sequence; When a positive integer  $k$  is assign to  $f(u)$ , it implies that  $P$  must have  $k$  vertices if  $T$  has unique path sequence and  $u$  is an end of a path  $P$  in some minimum path covering of  $T'$ . Hence, once  $A$  is assign to  $\ell(u)$  in Steps 4, 5 and 6, we take  $f(u) = 0$ . In Step 3, if  $T''$  is neither a path

nor a generalized star, by Theorem 18, the algorithm outputs “ $T^c$  has multiple island sequences”. In Steps 4, 5 and 6, the algorithm first checks two labels of each end of vines of  $S$  (if  $T''$  is a generalized star  $S$ ) or each leaf of path  $P_k$  (if  $T''$  is a path  $P_k$ ) to determine whether the algorithm output “ $T^c$  has multiple island sequence”; Then determine whether we need to change two labels of  $u$ . At last, let  $T = T'$  go back to Step 1. In Step 1, we consider a path  $P_k$  with  $k \geq 1$ . If there is a leaf or isolated vertex  $x$  with  $l(x) = A$  or  $0 < f(x) \neq k$ , it is obvious that  $T$  has no unique path sequence. In fact, since  $d'_T(r, v) = d_T(r, v)$  for each vertex  $v \in V(T')$ , where  $r$  is the selected leaf in Step 2, Step 2 need to perform only one time. Note that every vertex of  $T$  are used in a constant number in **Algorithm DUIS**. Hence, It is easy to see that **Algorithm DUIS** has complexity  $O(|V(T)|)$ .

**Theorem 19** *Algorithm DUIS can determine whether its complement  $T^c$  of a given tree  $T$  has unique island sequence in  $O(|V(T)|)$  time.*

## 5 Some invariants of graphs and their complements

In this section we extend some results in Section 3 to more general graphs.

In [2], Adams et. al determined the path covering number of connected non-cycle 2-sparse graphs. We now extend their result.

**Theorem 20** *Let  $G$  be a connected non-cycle graph with  $m \geq 1$  edges,  $n$  vertices,  $h$  heavy edges, and  $\ell$  leaves. If every heavy vertex in  $G$  is adjacent to at least three light vertices, then  $P(G) = \ell + m - h - n$ .*

**Proof.** The proof proceeds by induction on  $h$ , the number of heavy edges in  $G$ . If  $h = 0$ , then  $G$  is 2-sparse. By Theorem 6 (cf. Theorem 3.2 in [2]), We know that  $P(G) = \ell + m - n$ . Let us assume that  $h > 1$  and that the result holds for any connected non-cycle graph with  $k$  ( $1 \leq k < h$ ) heavy edges if its each heavy vertex has at least three light neighbors. Consider  $G$  a connected non-cycle graph with  $m$  edges,  $n$  vertices,  $h$  heavy edges, and  $\ell$  leaves. Assume that each heavy vertex in  $G$  has at least three light neighbors. Let  $e = uv$  be a heavy edge in  $G$  and let  $G_1, \dots, G_t$  be connected components of  $G - e$ , where  $1 \leq t \leq 2$ . By Lemma 10,

$P(G) = P(G - e) = \sum_{i=1}^t P(G_i)$ . Since  $u$  ( $v$ , respectively) is adjacent to at least three light vertices, each  $G_i$  is not a cycle and  $G_i$  satisfies the induction hypothesis, and hence  $P(G_i) = \ell_i + m_i - h_i - n_i$ , where  $m_i$  ( $n_i$ ,  $h_i$ ,  $\ell_i$ , respectively) is the number of edges (vertices, heavy vertices, leaves, respectively) in  $G_i$ . Note that the following equalities hold:

$$\begin{aligned} \sum_{i=1}^t \ell_i &= \ell & \text{and} & & \sum_{i=1}^t m_i &= m - 1; \\ \sum_{i=1}^t n_i &= n & \text{and} & & \sum_{i=1}^t h_i &= h - 1. \end{aligned}$$

Therefore,  $P(G) = \ell + m - h - n$  and the result follows.  $\blacksquare$

Similarly, the following result is a direct corollary of Theorem 20, Theorem 1 and Theorem 2.

**Corollary 21** *Let  $G$  be a connected non-cycle graph with  $m \geq 1$  edges,  $n$  vertices,  $h$  heavy edges, and  $\ell$  leaves. If every heavy vertex in  $G$  is adjacent to at least three light vertices and  $\ell + m \geq h + n + 2$ , then  $\lambda(G^c) = \ell + m - h - 2$  and  $\rho(G^c) = \ell + m - h - n - 1$ .*

A *complete graph*  $K_n$  is a graph of order  $n \geq 2$  in which every two vertices are adjacent. A vertex  $v \in V(G)$  is a *cut-vertex* if deleting  $v$  and all edges incident to it increases the number of connected components. A *block* of  $G$  is a maximal connected subgraph of  $G$  without cut-vertex. A *block graph* is a connected graph whose blocks are complete graphs. If every block is  $K_2$ , then it is a tree.

Given a nontrivial tree  $T$ ,  $\mathcal{G}(T)$  is defined as a family of block graphs obtained from  $T$  by expanding each edge of  $T$  into a complete graph of arbitrary order. It is obvious that  $T \in \mathcal{G}(T)$ . Figure 5 illustrates a graph  $G$  obtained from  $P_3$  by expanding edges  $e$ ,  $f$  into  $K_3$ ,  $K_4$ , respectively.

Now we give a result to extend the result in Theorem 5 (cf. Theorem 2.4 in [2]).

**Theorem 22** *Let  $T$  be a 2-sparse tree with  $\ell \geq 2$  leaves. Then  $P(G) = \ell - 1$  for any graph  $G \in \mathcal{G}(T)$ .*

**Proof.** Let  $G \in \mathcal{G}(T)$ . As  $G$  is an expansion of  $T$ , it is easy to see that  $P(G) \geq P(T) = \ell - 1$ . Consider an arbitrary minimum path covering of  $T$ , we will use this

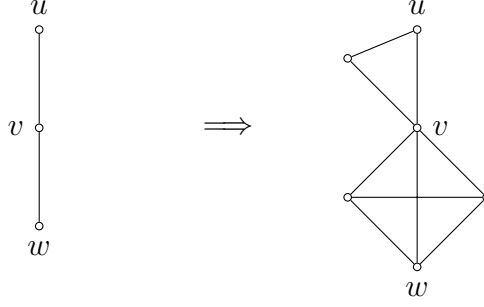


Figure 5

minimum path covering to construct a path covering of  $G$  with exactly  $\ell - 1$  paths, which will imply that  $P(G) = \ell - 1$ . Let  $e = uv$  be an arbitrary edge of  $T$ . Assume that the vertices of the block of  $G$  replacing  $e$  are  $u, x_1, \dots, x_t, v$  (For convenience, two cut-vertices in this block are still called as  $u, v$ ). We now construct a path covering of  $G$  as follows: If  $e$  is contained in a path  $P = \dots uv \dots$  in this minimum path covering of  $T$ , then we construct a path  $P' = \dots ux_1 \dots x_tv \dots$ ; If  $e$  is not contained in any path in this minimum path covering of  $T$ , by Lemma 7, one of  $u, v$  is a heavy vertex. Without loss of generality, we assume that  $v$  is a heavy vertex. As  $T$  is 2-sparse, we know that  $u$  is a light vertex in  $T$  and hence  $u$  is one end of a path, say  $P$ , in this minimum path covering of  $T$ . Then we construct a path  $P' = Px_1 \dots x_t$ . ■

Our final corollary below determines the  $\lambda$  and  $\rho$  of complements of graphs satisfying the condition of Theorem 22.

**Corollary 23** *Let  $T$  be a 2-sparse tree with  $\ell \geq 3$  leaves. Then  $\lambda(G^c) = n + \ell - 3$  and  $\rho(G^c) = \ell - 2$  for any  $G \in \mathcal{G}(T)$ , where  $n$  is the order of  $G$ .*

At the end of this section, we shall point out that it is interesting to establish the path covering numbers of general 2-sparse graphs.

## 6 Conclusions

In this paper, we determined the path covering number of some families of trees and tree-like graphs. Additionally, we determined  $\lambda$  and  $\rho$  for its complements. We also established a constructive characterization for trees whose complements admit unique

island sequences. A linear-time algorithm was also given to determine whether the complement of a given tree  $T$  admits unique island sequence. Hence, an open question in [12] was answered. Our work generalized most of results in [2]. We hope these results will be extended to include more general trees and graphs.

## References

- [1] S. S. Adams, M. Tesch, D. S. Troxell and C. Wheeland, On the hole index of  $L(2, 1)$ -labelings of  $r$ -regular graphs, *Discrete Appl. Math.* **155** (2007), 2391-2393.
- [2] S. S. Adams, A. Trazkovich, D. S. Troxell and B. Westgate, On island sequences of labelings with a condition at distance two, *Discrete Appl. Math.* **158** (2010), 1-7.
- [3] S. R. Arikati and C. P. Rangan, Linear algorithm for optimal path cover problem on interval graphs, *Inform. Process. Lett.* **35** (1990), 149-153.
- [4] F. T. Boesch, S. Chen and N. A. M. McHugh, On covering the points of a graph with point disjoint paths, in: A. Dold, B. Eckman (Eds.), *Graphs and Combinatorics*, in: *Lecture Notes in Math.*, vol. 406, Springer, Berlin, 1974, pp. 201-212.
- [5] F. T. Boesch and J. F. Gimpel, Covering the points of a digraph with point-disjoint paths and its application to code optimization, *J. Assoc. Comput.* **24** (1977), 192-198.
- [6] T. Calamoneri, The  $L(h, k)$ -labeling problem: A survey and annotated bibliography, *Comput. J.* **49** (2006), 585-608.
- [7] G. J. Chang and D. Kuo, The  $L(2, 1)$ -labeling on graphs, *SIAM J. Discrete Math.* **9** (1996), 309-316.
- [8] P. Damaschke, J. S. Deogun, D. Kratsch and G. Steiner, Finding hamiltonian paths in cocomparability graphs using the bump number algorithm, *Order* **8** (1992), 383-391.
- [9] P. C. Fishburn and F. S. Roberts, No-hole  $L(2, 1)$ -colorings, *Discrete Appl. Math.* **130** (2003), 513-519.



- [10] P. C. Fishburn and F. S. Roberts, Full color theorems for  $L(2, 1)$ -labelings, *SIAM J. Discrete Math.* **20** (2006), 428-443.
- [11] D. S. Franzblau and A. Raychaudhuri, Optimal hamiltonian completions and path covers for trees, and a reduction to maximum flow, *Anziam J.* **44** (2002), 193-204.
- [12] J. Georges and D. W. Mauro, On the structure of graphs with non-surjective  $L(2, 1)$ -labelings, *SIAM J. Discrete Math.* **19** (2005), 208-223.
- [13] J. Georges and D. W. Mauro, A note on collections of graphs with non-surjective lambda labelings, *Discrete Appl. Math.* **146** (2005), 92-98.
- [14] J. Georges, D. W. Mauro and M. Whittlesey, Relating path covering to vertex labelings with a condition at distance two, *Discrete Math.* **135** (1994), 103-111.
- [15] J. R. Griggs and R. K. Yeh, Labeling graphs with a condition at distance two, *SIAM J. Discrete Math.* **5** (1992), 586-595.
- [16] S. E. Goodman and S. T. Hedetniemi, On the hamiltonian completion problem, in: A. Dold, B. Eckman (Eds.), *Graphs and Combinatorics*, in: *Lecture Notes in Math.*, vol. 406, Springer, Berlin, 1974, pp. 262-272.
- [17] R. W. Huang and M. S. Chang, Solving the path cover problem on circular arc graphs by using an approximation algorithm, *Discrete Appl. Math.* **154** (2006), 76-105.
- [18] R. W. Huang and M. C. Chang, Finding a minimum path cover of a distance-hereditary graph in polynomial time, *Discrete Appl. Math.* **155** (2007), 2242-2256.
- [19] D. Král, R. Škrekovski and M. Tancer, Construction of large graphs with no optimal surjective  $L(2, 1)$ -labelings, *SIAM J. Discrete Math.* **20** (2006), 536-543.
- [20] R. Lin, S. Olariu and G. Pruesse, An optimal path cover algorithm for cographs, *Comput. Math. Appl.* **30** (1995), 75-83.
- [21] C. Lu, L. Chen and M. Zhai, Extremal problems on consecutive  $L(2, 1)$ -labelings, *Discrete Appl. Math.* **155** (2007), 1302-1313.
- [22] C. Lu and M. Zhai, An extremal problem on non-full colorable graphs, *Discrete Appl. Math.* **155** (2007), 2165-2173.

- [23] J. Misra and R. E. Tarjan, Optimal chain partitions of trees, *Inform. Process. Lett.* **4** (1975), 24-26.
- [24] S. Moran and Y. Wolfstahl, Optimal covering of cacti by vertex disjoint paths, *Theoret. Comput. Sci.* **84** (1991), 179-197.
- [25] S. C. Ntafos and S. L. Hakim, On path cover problems in digraphs and applications to program testing, *IEEE Trans. Software Eng.* **5** (1979), 520-529.
- [26] D. Sakai, Labeling chordal graphs with a condition at distance two, *SIAM J. Discrete Math.* **7** (1994), 133-140.
- [27] P. J. Slater, Path coverings of the vertices of a tree, *Discrete Math.* **25** (1979), 65-74.
- [28] R. Srikant, R. Sundaram, K. S. Singh and C.P. Rangan, Optimal path cover problem on block graphs and bipartite permutation graphs, *Theoret. Comput. Sci.* **115** (1993), 351-357.
- [29] J. Van den heuvel, R. A. Leese and M. A. Shepherd, Graph labeling and radio channel assignment, *J. Graph Theory* **29** (1998), 263-283.
- [30] M. Whittlesey, J. Georges and D. W. Mauro, On the  $\lambda$ -number of  $Q_n$  and related graphs, *SIAM J. Discrete Math.* **8** (1995), 499-506.
- [31] P. K. Wong, Optimal path cover problem on block graphs, *Theoret. Comput. Sci.* **225** (1999), 163-169.
- [32] J. H. Yan, The path partition and related problems, Ph. D. Thesis, Dept. Applied Math., National Chiao Tung Univ., Hsingchu, Taiwan, 1994.
- [33] J. H. Yan and G. J. Chang, The path-partition problem in block graphs, *Inform. Process. Lett.* **52** (1994), 317-322.
- [34] R. K. Yeh, A survey on labeling graphs with a condition at distance two, *Discrete Math.* **306** (2006), 1217-1231.
- [35] S. Zhou, Labelling Cayley graphs on Abelian groups, *SIAM J. Discrete math.* **19** (2006), 985-1003.
- [36] S. Zhou, Distance labelling problems for hypercubes and Hamming graphs-a survey, *Elec. Notes Discrete Math.* **28** (2007), 527-534.